

BOREL EQUIVALENCE RELATIONS IN THE SPACE OF BOUNDED OPERATORS

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ABSTRACT. We consider various notions of equivalence in the space of bounded operators on a Hilbert space, in particular modulo finite rank, modulo Schatten p -class, and modulo compact. Using Hjorth’s theory of turbulence, the latter two are shown to be not classifiable by countable structures, while the first is not reducible to the orbit equivalence relation of any Polish group action. The results for modulo finite rank and modulo compact operators are also shown for the restrictions of these equivalence relations to the space of projection operators.

1. INTRODUCTION

A fundamental problem in the theory of operators on an infinite dimensional separable complex Hilbert space is to classify a collection of operators up to some notion of equivalence, a classical example being the following:

Theorem (Weyl–von Neumann [18]). *For T and S bounded self-adjoint operators on a Hilbert space as above, the following are equivalent:*

- (i) *T and S are unitarily equivalent modulo compact, i.e., there is a compact operator K and a unitary operator U such that $UTU^* - S = K$.*
- (ii) *T and S have the same essential spectrum.*

That is, bounded self-adjoint operators are completely classified up to unitary equivalence modulo compact by their essential spectra (for a definition see p. 30 in [15]).

The modern theory of Borel equivalence relations affords us a general framework for such results. Given a space X of objects, and an equivalence relation E on X , completely classifying the elements of X up to E -equivalence amounts to finding another space Y with equivalence relation F , and specifying a map $f : X \rightarrow Y$ such that

$$xEy \iff f(x)Ff(y),$$

for all $x, y \in X$. The spaces and equivalence relations should be “reasonably definable”, in the sense that the former are Polish (or more generally, standard Borel) and the latter Borel. Enforcing that the classifying map f is Borel captures that idea that f is “computing” an invariant for the objects in X . Such a map is called

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a *Borel reduction* of E to F , and its existence or non-existence allows us to compare the complexity of such equivalence relations. The “simplest” Borel equivalence relations are those given by equality on Polish spaces, and are said to be *smooth*.

Recasting the motivating problem into this setting requires specifying a Polish or Borel structure on the collection of operators of interest, verifying that the notion of equivalence is Borel, and reducing the equivalence relation to another, preferably well-understood, equivalence relation. In the setting of the Weyl–von Neumann theorem above, we have:

Theorem (Ando–Matsuzawa [1]). *The map $T \mapsto \sigma_{\text{ess}}(T)$ is a Borel function from the space of bounded self-adjoint operators to the Effros Borel space of closed subsets of \mathbb{R} . In particular, unitary equivalence modulo compact of bounded self-adjoint operators is smooth.*

In contrast, many natural equivalence relations on classes of operators are not smooth. In fact, they exhibit a very strong form of non-classifiability; they cannot be reduced to the isomorphism relation on any class of countable algebraic or relational structures, e.g., groups, rings, graphs, etc. Such equivalence relations are said to be *not classifiable by countable structures*. The method used to exhibit this property is Hjorth’s theory of turbulence [9]. A relevant example is given by:

Theorem (Kechris–Sofronidis [13]). *Unitary equivalence of self-adjoint (or unitary) operators is not classifiable by countable structures.*

We also have the following for (densely defined) unbounded operators:

Theorem (Ando–Matsuzawa [1]). *Unitary equivalence modulo compact of unbounded self-adjoint operators is not classifiable by countable structures.*

In this article, we present classification and non-classification results for collections of operators, focusing on equivalence relations induced by ideals of compact operators. The paper is arranged as follows:

In §2, we review the relevant theory of bounded operators and Borel equivalence relations.

In §3, we consider equivalence relations on the sequence spaces $\mathbb{R}^{\mathbb{N}}$, $[0, 1]^{\mathbb{N}}$ and $\mathbb{T}^{\mathbb{N}}$, arising from actions of c_0 and ℓ^p for $1 \leq p < \infty$. We show that equality modulo c_0 and ℓ^p on $[0, 1]^{\mathbb{N}}$ are Borel bireducible with orbit equivalence relations of turbulent actions of these spaces on $\mathbb{T}^{\mathbb{N}}$.

In §4, we describe the relevant Borel and Polish structures on collections of operators, and in §5, establish that equivalence of operators modulo compact and modulo Schatten p -class are not classifiable by countable structures. We also show that equivalence of operators modulo finite rank is not Borel reducible to the orbit equivalence relation of any Polish group action.

In §6, we restrict our attention to projection operators. After considering equivalence relations given by rank, we turn to the restrictions of modulo finite rank and modulo compact to the projections. The latter provides an alternate view of the *projections in the Calkin algebra*, a structure that has also been studied in combinatorial set theory, see [20] and [21]. We show that equivalence of projections modulo finite rank is not Borel reducible to the orbit equivalence relation of any Polish group action, and that equivalence modulo compact is not classifiable by countable structures.

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2. PRELIMINARIES

2.1. Bounded operators on Hilbert spaces. Throughout, we fix an infinite dimensional separable complex Hilbert space H , with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(H)$ denote the set of all bounded operators on H . A standard reference for the relevant material on operators is [17].

Endowed with the operator norm $\| \cdot \|$, $\mathcal{B}(H)$ is a Banach space, and we refer to this topology as the *norm topology*. The *strong operator topology* is the (weaker) topology induced by the family of seminorms $T \mapsto \|Tv\|$ for $v \in H$, or equivalently, the topology of pointwise convergence on H .

We denote by $T \mapsto T^*$ the map taking an operator to its adjoint, which gives $\mathcal{B}(H)$ (with the operator norm) the structure of a C^* -algebra. An operator $T \in \mathcal{B}(H)$ is *self-adjoint* if $T = T^*$, while a (necessarily self-adjoint) operator T is *positive* if $\langle Tv, v \rangle \geq 0$ for all $v \in H$. To each operator $T \in \mathcal{B}(H)$, there is a unique positive operator $|T|$ satisfying $\| |T|v \| = \|Tv\|$ for all $v \in H$, and $|T|^2 = T^*T$ (3.2.17 in [17]). An operator $U \in \mathcal{B}(H)$ is *unitary* if $UU^* = U^*U = I$, where I is the identity operator.

An operator $P \in \mathcal{B}(H)$ is a *projection* if $P^2 = P^* = P$. Equivalently, P is the orthogonal projection onto a closed subspace (namely, $\text{ran}(P)$) of H (3.2.13 in [17]). Every projection is positive with $\|P\| = 1$ whenever $P \neq 0$. We denote the set of projections by $\mathcal{P}(H)$. If P is a projection, and $\{f_k : k \in \mathbb{N}\}$ an orthonormal basis for $\text{ran}(P)$, then for $v \in H$,

$$Pv = \sum_{k=0}^{\infty} \langle v, f_k \rangle f_k.$$

An operator $T \in \mathcal{B}(H)$ is *diagonal* with respect to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H if there is a bounded sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that

$$Tv = \sum_{n=0}^{\infty} \lambda_n \langle v, e_n \rangle e_n$$

for all $v \in H$.

An operator $T \in \mathcal{B}(H)$ is *compact* if the image of the closed unit ball of H under T is compact. The set of compact operators is denoted by $\mathcal{K}(H)$. T is *finite rank* if $\text{rank}(T) = \dim(\text{ran}(T)) < \infty$, and the set of finite rank operators is denoted by $\mathcal{B}_f(H)$. Clearly, $\mathcal{B}_f(H) \subseteq \mathcal{K}(H)$, and one can show that an operator is compact if and only if it is a norm limit of finite rank operators (3.3.3 in [17]). The following characterizes which diagonal operators are compact:

Proposition 2.1 (3.3.5 in [17]). *If $T \in \mathcal{B}(H)$ is diagonal with respect to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$, say $Tv = \sum_{n=0}^{\infty} \lambda_n \langle v, e_n \rangle e_n$ for all $v \in H$, then T is compact if and only if $\lim_{n \rightarrow \infty} \lambda_n = 0$.*

It is easy to check that $\mathcal{K}(H)$ is a norm-closed, self-adjoint ideal in $\mathcal{B}(H)$, and the corresponding quotient $\mathcal{B}(H)/\mathcal{K}(H)$ is called the *Calkin algebra*.

If T is a positive operator and $\{e_n : n \in \mathbb{N}\}$ an orthonormal basis of H , the *trace* of T is

$$\operatorname{tr}(T) = \sum_{n=0}^{\infty} \langle Te_n, e_n \rangle.$$

This value (which may be ∞) is independent of the choice of basis (3.4.4 in [17]). For $1 \leq p < \infty$, the *Schatten p -class* $\mathcal{B}^p(H)$ is defined to be all operators $T \in \mathcal{B}(H)$ such that $\operatorname{tr}(|T|^p) < \infty$. For $p = 1$, $\mathcal{B}^1(H)$ is the set of *trace-class operators*, and for $p = 2$, $\mathcal{B}^2(H)$ is the set of *Hilbert-Schmidt operators*. Each $\mathcal{B}^p(H)$ is a self-adjoint ideal in $\mathcal{B}(H)$ (which fails to be norm-closed), and $\mathcal{B}_f(H) \subsetneq \mathcal{B}^p(H) \subsetneq \mathcal{K}(H)$.

2.2. Borel equivalence relations. A *Polish space* is a separable and completely metrizable topological space X , and an equivalence relation E on X is *Borel* if $\{(x, y) \in X^2 : xEy\}$ is a Borel subset of X^2 . Given equivalence relations E and F on Polish spaces X and Y , respectively, a map $f : X \rightarrow Y$ is a *Borel reduction* of E to F if f is Borel measurable, and

$$xEy \Leftrightarrow f(x)Ff(y)$$

for all $x, y \in X$. Equivalently, f is a Borel map which descends to a well-defined injection $X/E \rightarrow Y/F$. In this case, we say that E is *Borel reducible* to F , and write $E \leq_B F$. If f is injective, we say that f is a *Borel embedding* of E into F , and write $E \sqsubseteq_B F$. If $E \leq_B F$ and $F \leq_B E$, we write $E \equiv_B F$, and say that E and F are *Borel bireducible*. Intuitively, $E \leq_B F$ means that classifying elements of Y up to F is at least as complicated as classifying elements of X up to E , as any classification of the former yields one for the latter. $E \equiv_B F$ means that they are of equal complexity.

Example 2.2. If X is a Polish space, we denote by $\Delta(X)$ the equality relation on X . $\Delta(X)$ is a closed, and thus Borel, subset of X^2 .

Example 2.3. Identifying $2 = \{0, 1\}$, with the discrete topology, the Borel equivalence relation E_0 is defined on $2^{\mathbb{N}}$ by

$$(x_n)_n E_0 (y_n)_n \Leftrightarrow \exists m \forall n \geq m (x_n = y_n).$$

Example 2.4. The Borel equivalence relation E_1 is defined on $\mathbb{R}^{\mathbb{N}}$ by

$$(x_n)_n E_1 (y_n)_n \Leftrightarrow \exists m \forall n \geq m (x_n = y_n).$$

We say that a Borel equivalence relation E on a Polish space X is *smooth* if $E \leq_B \Delta(Y)$ for some Polish space Y . Since all uncountable Polish spaces are Borel isomorphic (Corollary 1.3.8 in [6]), smooth equivalence relations are exactly those which admit complete classification by real numbers. It is well-known that E_0 is not smooth (cf. §6.1 in [6]), and by a theorem of Harrington–Kechris–Louveau [7], is a canonical obstruction to smoothness.

A *Polish group* G is a topological group which has a Polish topology. If X is a Polish space, and G acts continuously on X , i.e., the map $G \times X \rightarrow X : (g, x) \mapsto g \cdot x$ is continuous, then we say that X is a *Polish G -space*, and denote by E_G (or sometimes E/G) the orbit equivalence relation

$$x E_G y \Leftrightarrow \exists g \in G (g \cdot x = y).$$

This equivalence relation is not Borel in general (see §9.4 of [6] for examples).

A group with a given Borel structure (e.g., a Borel subgroup of a Polish group) is *Polishable* if it can be endowed with a Polish group topology having the same

Borel structure. It is easy to check that the orbit equivalence relation induced by the translation action of a Polishable (or Borel) subgroup of a Polish group is Borel.

The following theorem shows that E_1 is an obstruction to classification by orbits of Polish group actions.

Theorem 2.5 (Kechris–Louveau [12]). *Let G be a Polish group, and X a Polish G -space. Then, $E_1 \not\leq_B E_G^X$.*

The isomorphism relation on the class of countable structures of a first-order theory, e.g., groups, rings, graphs, etc, can be represented as the orbit equivalence relation of a Polish G -space (cf. Ch. 11 of [6]). If an equivalence relation is Borel reducible to such a relation, we say that it is *classifiable by countable structures*. Hjorth [9] isolated a dynamical property of Polish G -spaces, called *turbulence*, which implies that the corresponding orbit equivalence relation resists such classification.

Let X be a Polish G -space. For $U \subseteq X$ open, and $V \subseteq G$ a symmetric open neighborhood of the identity, the (U, V) -local orbit $\mathcal{O}(x, U, V)$ of a point $x \in U$ is the collection of all points $y \in U$ such that there are $g_0, \dots, g_k \in V$ and $x = x_0, \dots, x_{k+1} = y \in U$ with $x_{i+1} = g_i \cdot x_i$ for $i \leq k$. For such an X and G , we say that the action of G is *turbulent* if every orbit is dense, every orbit is meager, and every (U, V) -local orbit is *somewhere dense*, i.e., for every such U, V and x , $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

Theorem 2.6 (Hjorth [9]). *Let X be a Polish G -space. If the action of G is turbulent, then E_G is not classifiable by countable structures.*

3. EXAMPLES AND RESTRICTIONS OF TURBULENT ACTIONS

The following examples of turbulent actions are particularly relevant for studying equivalence relations arising in analysis.

Example 3.1. We say that a subgroup G of the additive group $\mathbb{R}^{\mathbb{N}}$ is *strongly dense* if for every finite sequence (x_0, \dots, x_n) of real numbers, there is a $y = (y_0, y_1, \dots) \in G$ such that $y_i = x_i$ for $i \leq n$. If G is a proper, Polishable, and strongly dense subgroup of $\mathbb{R}^{\mathbb{N}}$, then the translation action of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent (Proposition 3.25 in [9]). Following [3] and [8], which study the example subgroups c_0 and ℓ^p for $1 \leq p < \infty$, we denote the corresponding equivalence relation by $\mathbb{R}^{\mathbb{N}}/G$.

Example 3.2. Let X be a separable Frechet space. If Y is a proper, Polishable, dense subspace of X , then the translation action of Y on X is turbulent (see p. 35 in [11]). Examples of such pairs (X, Y) include $(\mathbb{R}^{\mathbb{N}}, c_0)$ and $(\mathbb{R}^{\mathbb{N}}, \ell^p)$, as well as $(L^p([0, 1]), C([0, 1]))$ and (c_0, ℓ^p) for $1 \leq p < \infty$.

For our applications, we consider the restrictions of these equivalence relations to the subsets $[0, 1]^{\mathbb{N}}$ and c_0 of $\mathbb{R}^{\mathbb{N}}$. We denote by $[0, 1]^{\mathbb{N}}/c_0$ the restriction of $\mathbb{R}^{\mathbb{N}}/c_0$ to the subset $[0, 1]^{\mathbb{N}}$, similarly for $[0, 1]^{\mathbb{N}}/\ell^p$ and c_0/ℓ^p . It is evident that these are Borel equivalence relations, and that $[0, 1]^{\mathbb{N}}/c_0 \subseteq_B \mathbb{R}^{\mathbb{N}}/c_0$, $[0, 1]^{\mathbb{N}}/\ell^p \subseteq_B \mathbb{R}^{\mathbb{N}}/\ell^p$ and $c_0/\ell^p \subseteq_B \mathbb{R}^{\mathbb{N}}/\ell^p$ for $1 \leq p < \infty$, via the inclusion maps. Moreover, each of these is known to be not classifiable by countable structures; this is evident for c_0/ℓ^p by Example 3.2, and can be shown for $[0, 1]^{\mathbb{N}}/c_0$ and $[0, 1]^{\mathbb{N}}/\ell^p$ using Lemma 3.1 of [14]. In fact, we have the following, part (a) of which is essentially due to Oliver [16]:

Theorem 3.3 (Lemma 6.2.2 in [10]). (a) $\mathbb{R}^{\mathbb{N}}/c_0 \equiv_B [0, 1]^{\mathbb{N}}/c_0$.

(b) For $1 \leq p < \infty$, $\mathbb{R}^{\mathbb{N}}/\ell^p \equiv_B [0, 1]^{\mathbb{N}}/\ell^p$.

By considering equivalence relations on $\mathbb{T}^{\mathbb{N}}$ induced by actions of c_0 and ℓ^p , where \mathbb{T} is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, we establish an alternate justification for the non-classifiability of $[0, 1]^{\mathbb{N}}/c_0$ and $[0, 1]^{\mathbb{N}}/\ell^p$, and give additional equivalents up to Borel bireducibility.

As before, a subgroup G of $\mathbb{T}^{\mathbb{N}}$ is *strongly dense* if for all finite sequences (z_0, \dots, z_n) of unit complex numbers, there is a $g = (g_0, g_1, \dots) \in G$ such that $g_i = z_i$ for $i \leq n$. The proof of the following is modeled on the corresponding result for strongly dense subgroups of $\mathbb{R}^{\mathbb{N}}$.

Proposition 3.4. *If G is a proper, Polishable, and strongly dense subgroup of $\mathbb{T}^{\mathbb{N}}$, then the translation action of G on $\mathbb{T}^{\mathbb{N}}$ is turbulent.*

Proof. Let G be as described. Clearly every orbit is dense. That G , and hence every orbit, is meager, follows from G being proper and Borel, by Pettis' Theorem (Theorem 2.3.2 in [6]).

It remains to verify that every local orbit is somewhere dense. Let $U \subseteq \mathbb{T}^{\mathbb{N}}$ be open, and $x \in U$. We may assume that the first m factors of U are arcs about x_j , for $j < m$, and the remaining factors are all of \mathbb{T} . Let $V \subseteq G$ be an open neighborhood of $1 = (1, 1, 1, \dots)$. Take $y \in U$ arbitrary, and let $U_0 \subseteq U$ be an open neighborhood of y whose first $M \geq m$ factors are neighborhoods of the corresponding coordinates of y , the rest being all of \mathbb{T} . We claim that $U_0 \cap \mathcal{O}(x, U, V) \neq \emptyset$, showing that $\overline{\mathcal{O}(x, U, V)} = U$.

Consider the projection $\pi_M : G \rightarrow \mathbb{T}^M : g \mapsto g \upharpoonright M$. Since G is Polishable, strongly dense, and π_M is a homomorphism, Pettis' Theorem implies that π_M is both continuous and open. Let $W = \pi_M(V)$.

For $j < M$, pick $\varphi_j \in (-\pi, \pi]$ such that $e^{i\varphi_j} x_j = y_j$, and moreover, the arc $[0, 1] \rightarrow \mathbb{T} : t \mapsto e^{it\varphi_j} x_j$ is entirely contained in the j th factor of U . This can be done by our assumptions on U . Pick an integer $N \geq 1$ large enough so that $w = (e^{i\varphi_0/N}, \dots, e^{i\varphi_M/N}) \in W$, and let $g \in V$ be such that $\varphi_m(g) = w$. Then, each of x, gx, g^2x, \dots, g^Nx is in U , and $g^Nx \in U_0 \cap \mathcal{O}(x, U, V)$. \square

For G a subgroup of $\mathbb{R}^{\mathbb{N}}$, consider the map

$$\varphi_G : G \rightarrow \mathbb{T}^{\mathbb{N}} : (\alpha_n)_n \mapsto (e^{i\alpha_n})_n.$$

Lemma 3.5. *For G a Polishable subgroup of $\mathbb{R}^{\mathbb{N}}$ and φ_G as above:*

- (a) φ_G is a continuous group homomorphism.
- (b) $\text{ran}(\varphi_G)$ is a Polishable subgroup of $\mathbb{T}^{\mathbb{N}}$.
- (c) If G is strongly dense in $\mathbb{R}^{\mathbb{N}}$, then $\text{ran}(\varphi_G)$ is strongly dense in $\mathbb{T}^{\mathbb{N}}$.

Proof. (a) φ_G is clearly a homomorphism and Borel on G . By compatibility of the topology, and Pettis' Theorem (Theorem 2.3.3 in [6]), it is continuous.

(b) Let $K_G = \ker(\varphi_G)$, a closed subgroup of G . Let τ be the Polish group topology on $\text{ran}(\varphi_G)$ making the induced map $G/K_G \rightarrow \text{ran}(\varphi_G)$ a topological group isomorphism. Let $\varphi = \varphi_{\mathbb{R}^{\mathbb{N}}} : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{T}^{\mathbb{N}}$, and $K = \ker(\varphi)$. Then, $\iota : g + K_G \mapsto g + K$ is a well-defined, injective, continuous group homomorphism making the following

diagram commute:

$$\begin{array}{ccc}
 G & \xrightarrow{\subseteq} & \mathbb{R}^{\mathbb{N}} \\
 \downarrow & & \downarrow \\
 G/K_G & \xrightarrow{\iota} & \mathbb{R}^{\mathbb{N}}/K \\
 \cong \downarrow & & \downarrow \cong \\
 \text{ran}(\varphi_G) & \xrightarrow{\subseteq} & \mathbb{T}^{\mathbb{N}}
 \end{array}$$

If $B \subseteq G/K_G$ is Borel, then $\iota(B)$ is Borel in $\mathbb{R}^{\mathbb{N}}/K$ being a continuous injective image of a Borel set. Passing through the topological isomorphisms $G/K_G \cong \text{ran}(\varphi_G)$ and $\mathbb{R}^{\mathbb{N}}/K \cong \mathbb{T}^{\mathbb{N}}$, we have that every τ -Borel subset of $\text{ran}(\varphi_G)$ is Borel in $\mathbb{T}^{\mathbb{N}}$, verifying compatibility of the Borel structure.

(c) This claim is obvious. \square

The subgroups of $\mathbb{T}^{\mathbb{N}}$ in which we are interested are those arising as $\text{ran}(\varphi_G)$, where G is one of c_0 or ℓ^p , for $1 \leq p < \infty$. Denote by

$$G_0 = \text{ran}(\varphi_{c_0}) \quad \text{and} \quad G_p = \text{ran}(\varphi_{\ell^p}).$$

Observe that $G_p \subseteq G_0$, and G_0 is proper in $\mathbb{T}^{\mathbb{N}}$. By the previous lemma and Proposition 3.4, G_0 and G_p are strongly dense, Polishable subgroups of $\mathbb{T}^{\mathbb{N}}$ which act turbulently on $\mathbb{T}^{\mathbb{N}}$ by translation. These actions are orbit equivalent to the actions of c_0 and ℓ^p given by

$$(\alpha_n)_n \cdot (e^{i\theta_n})_n = e^{i(\theta_n + \alpha_n)},$$

for $(\alpha_n)_n$ in c_0 and ℓ^p , respectively. In what follows, let ρ denote the *arc length metric* on \mathbb{T} .

Lemma 3.6. (a) *For $z, w \in \mathbb{T}^{\mathbb{N}}$, the following are equivalent:*

$$\begin{aligned}
 zw^{-1} \in G_0 &\Leftrightarrow \lim_n \rho(z_n, w_n) = 0 \\
 &\Leftrightarrow \lim_n z_n w_n^{-1} = 1 \\
 &\Leftrightarrow \lim_n |\text{Re}(z_n - w_n)| + |\text{Im}(z_n - w_n)| = 0.
 \end{aligned}$$

In particular, $G_0 = \{z \in \mathbb{T}^{\mathbb{N}} : \lim_n z_n = 1\}$.

(b) *For $1 \leq p < \infty$, and $z, w \in \mathbb{T}^{\mathbb{N}}$,*

$$\begin{aligned}
 zw^{-1} \in G_p &\Leftrightarrow \sum_{n=0}^{\infty} \rho(z_n, w_n)^p < \infty \\
 &\Leftrightarrow \sum_{n=0}^{\infty} |z_n w_n^{-1} - 1|^p < \infty \\
 &\Leftrightarrow \sum_{n=0}^{\infty} |\text{Re}(z_n - w_n)|^p + \sum_{n=0}^{\infty} |\text{Im}(z_n - w_n)|^p < \infty
 \end{aligned}$$

In particular, $G_p = \{z \in \mathbb{T}^{\mathbb{N}} : \sum_{n=0}^{\infty} |z_n - 1|^p < \infty\}$.

Proof. Let d_p denote the p -norm metrics on \mathbb{R}^2 for $1 \leq p < \infty$. One can show that all of these metrics are Lipschitz equivalent to ρ when restricted to \mathbb{T} (embedded in \mathbb{R}^2 in the usual way).

To prove (a), the first and second equivalences are obvious, while the third is immediate from the Lipschitz equivalence of ρ and d_1 on \mathbb{T} .

For (b), fix $1 \leq p < \infty$. Again, the first equivalence is obvious, and letting $z, w \in \mathbb{T}^{\mathbb{N}}$, we have

$$\sum_{n=0}^{\infty} \rho(z_n, w_n)^p < \infty \Leftrightarrow \sum_{n=0}^{\infty} \rho(z_n w_n^{-1}, 1)^p < \infty \Leftrightarrow \sum_{n=0}^{\infty} |z_n w_n^{-1} - 1|^p < \infty,$$

by the invariance of ρ , and the Lipschitz equivalence of ρ and d_2 on \mathbb{T} , proving the second. For the remaining equivalence, suppose that $zw^{-1} \in G_p$. Since $(|\operatorname{Re}(z_n - w_n)|^p + |\operatorname{Im}(z_n - w_n)|^p)^{1/p} \leq M\rho(z_n, w_n)$ for some constant M (which depends only on p), we have

$$\sum_{n=0}^{\infty} |\operatorname{Re}(z_n - w_n)|^p + \sum_{n=0}^{\infty} |\operatorname{Im}(z_n - w_n)|^p \leq M \sum_{n=0}^{\infty} \rho(z_n, w_n)^p < \infty.$$

Conversely, if $\sum_{n=0}^{\infty} |\operatorname{Re}(z_n - w_n)|^p + \sum_{n=0}^{\infty} |\operatorname{Im}(z_n - w_n)|^p < \infty$, then by Minkowski's Inequality, we have

$$\left(\sum_{n=0}^{\infty} d_1(z_n, w_n)^p \right)^{1/p} \leq \left(\sum_{n=0}^{\infty} |\operatorname{Re}(z_n - w_n)|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |\operatorname{Im}(z_n - w_n)|^p \right)^{1/p} < \infty.$$

Since ρ is Lipschitz equivalent to d_1 on \mathbb{T} , it follows that $\sum_{n=0}^{\infty} |\rho(z_n, w_n)|^p < \infty$, and $zw^{-1} \in G_p$. \square

Our goal for the remainder of this section is to show:

Theorem 3.7. (a) $[0, 1]^{\mathbb{N}}/c_0 \equiv_B \mathbb{T}^{\mathbb{N}}/G_0$.

(b) For $1 \leq p < \infty$, $[0, 1]^{\mathbb{N}}/\ell^p \equiv_B \mathbb{T}^{\mathbb{N}}/G_p$.

This result will follow by a series of three lemmas. Fix $1 \leq p < \infty$.

Lemma 3.8. (a) $[0, 1]^{\mathbb{N}}/c_0 \subseteq_B \mathbb{T}^{\mathbb{N}}/G_0$.

(b) $[0, 1]^{\mathbb{N}}/\ell^p \subseteq_B \mathbb{T}^{\mathbb{N}}/G_p$.

Proof. Both embeddings will be witnessed by the map:

$$f : [0, 1]^{\mathbb{N}} \rightarrow \mathbb{T}^{\mathbb{N}} : (\alpha_n)_n \mapsto (e^{i\pi/2(\alpha_n)})_n.$$

It is easy to see that this map is a continuous injection. By Lemma 3.6(a), if $\alpha, \beta \in [0, 1]^{\mathbb{N}}$, then

$$\alpha - \beta \in c_0 \Leftrightarrow \lim_n e^{i\pi/2(\alpha_n - \beta_n)} = 1 \Leftrightarrow f(\alpha)f(\beta)^{-1} \in G_0$$

establishing (a). (b) is shown similarly, using Lemma 3.6(b). \square

For G a subgroup of $\mathbb{R}^{\mathbb{N}}$, let $([0, 1]^{\mathbb{N}})^2/G \times G$ denote the equivalence relation E on $([0, 1]^{\mathbb{N}})^2$ given by

$$xEy \Leftrightarrow x - y \in G \times G,$$

for $x, y \in ([0, 1]^{\mathbb{N}})^2$.

Lemma 3.9. (a) $([0, 1]^{\mathbb{N}})^2/c_0 \times c_0 \equiv_B [0, 1]^{\mathbb{N}}/c_0$.

(b) $([0, 1]^{\mathbb{N}})^2 / \ell^p \times \ell^p \equiv_B [0, 1]^{\mathbb{N}} / \ell^p$.

Proof. Again, we will witness both embeddings by the same map:

$$f : ([0, 1]^{\mathbb{N}})^2 \rightarrow [0, 1]^{\mathbb{N}} : ((\alpha_n^0)_n, (\alpha_n^1)_n) \mapsto (\alpha_0^0, \alpha_0^1, \alpha_1^0, \alpha_1^1, \dots).$$

It is easy to see that this map is a homeomorphism. For $\alpha = ((\alpha_n^0)_n, (\alpha_n^1)_n)$ and $\beta = ((\beta_n^0)_n, (\beta_n^1)_n)$ in $([0, 1]^{\mathbb{N}})^2$, we have that

$$\alpha - \beta \in c_0 \times c_0 \iff \lim_{n \rightarrow \infty} (\alpha_n^0 - \beta_n^0) = 0 \text{ and } \lim_{n \rightarrow \infty} (\alpha_n^1 - \beta_n^1) = 0,$$

and so $\alpha - \beta \in c_0 \times c_0$ if and only if $f(\alpha) - f(\beta) \in c_0$, proving (a). (b) is proved similarly. \square

Lemma 3.10. (a) $\mathbb{T}^{\mathbb{N}} / G_0 \subseteq_B ([0, 1]^{\mathbb{N}})^2 / c_0 \times c_0$.

(b) $\mathbb{T}^{\mathbb{N}} / G_p \subseteq_B ([0, 1]^{\mathbb{N}})^2 / \ell^p \times \ell^p$.

Proof. We will show that $\mathbb{T}^{\mathbb{N}} / G_0$ and $\mathbb{T}^{\mathbb{N}} / G_p$ are continuously embeddable into $([-1, 1]^{\mathbb{N}})^2 / c_0 \times c_0$ and $([-1, 1]^{\mathbb{N}})^2 / \ell^p \times \ell^p$, respectively, which suffices since the latter are clearly continuously biembeddable with $([0, 1]^{\mathbb{N}})^2 / c_0 \times c_0$ and $([0, 1]^{\mathbb{N}})^2 / \ell^p \times \ell^p$. Again, we will use the same map to witness both:

$$f : \mathbb{T}^{\mathbb{N}} \rightarrow ([-1, 1]^{\mathbb{N}})^2 : (z_n)_n \mapsto ((\operatorname{Re} z_n)_n, (\operatorname{Im} z_n)_n).$$

Intuitively, we are simply placing each factor of $\mathbb{T}^{\mathbb{N}}$ into the plane in the obvious way. This map is clearly a continuous injection. Let $z, w \in \mathbb{T}^{\mathbb{N}}$. By Lemma 3.6(a), we have that

$$\begin{aligned} z w^{-1} \in G_0 &\iff \lim_n |\operatorname{Re}(z_n - w_n)| = 0 \text{ and } \lim_n |\operatorname{Im}(z_n - w_n)| = 0 \\ &\iff f(z) - f(w) \in c_0 \times c_0, \end{aligned}$$

showing that f is a reduction witnessing (a). (b) is proved similarly. \square

4. TOPOLOGY AND BOREL STRUCTURES IN $\mathcal{B}(H)$

In order to study Borel equivalence relations on $\mathcal{B}(H)$ or its subsets, we must endow them with a Polish or standard Borel structure. Most of the results in this section are routine, and can be found in [17].

Observe that the norm topology on $\mathcal{B}(H)$ is *not* Polish; it contains discrete subsets of size 2^{\aleph_0} (given an orthonormal basis $\{e_n : n \in \mathbb{N}\}$, consider the family of projections P_x onto $\operatorname{span}\{e_n : n \in x\}$, for $x \subseteq \mathbb{N}$). Instead, we focus on the strong operator topology.

Proposition 4.1 (4.6.2 in [17]). *$\mathcal{B}(H)_{\leq 1}$ is closed and Polish in the strong operator topology.*

Corollary 4.2. *$\mathcal{B}(H)$ is a standard Borel space with respect to the Borel structure generated by the strong operator topology.*

Proof. A countable union of standard Borel spaces is standard Borel, and clearly $\mathcal{B}(H) = \bigcup_{n \geq 1} n\mathcal{B}(H)_{\leq 1}$. \square

All references to Borel subsets of (or functions on) $\mathcal{B}(H)$ will be with respect to this Borel structure. We caution that $\mathcal{B}(H)$ is not Polish in the strong operator topology (it is not even metrizable, see E 4.6.4 in [17]), nor is it Polishable with this Borel structure. This follows from the fact that ℓ^∞ is not Polishable as a subspace of $\mathbb{R}^{\mathbb{N}}$ (Lemma 9.3.3 in [6]), and sits as a strongly closed subspace of $\mathcal{B}(H)$.

In order to show that certain subspaces of $\mathcal{B}(H)$ are Borel, we need to verify that certain maps on $\mathcal{B}(H)$ are Borel. To simplify what follows, it is useful to reference the *weak operator topology*, a topology which is weaker than the strong operator topology, but generates the same Borel structure (for a definition, see §4.6 in [17]). We caution that the adjoint operation is not strongly continuous on all of $\mathcal{B}(H)$ (E 4.6.1 in [17]), while multiplication is not jointly strongly continuous (cf. 4.6.1 in [17]).

Lemma 4.3 (cf. 4.6.1 in [17]). (a) *The adjoint operation is weakly continuous on $\mathcal{B}(H)$. In particular, $*$ is Borel.*

(b) *Multiplication of operators is strongly continuous when restricted to $B \times \mathcal{B}(H) \rightarrow \mathcal{B}(H)$, where B is any norm bounded subset of $\mathcal{B}(H)$. In particular, multiplication is Borel.*

Lemma 4.4. (a) *The set of all self-adjoint operators $\mathcal{B}(H)_{sa}$ is strongly closed in $\mathcal{B}(H)$, and thus Borel.*

(b) *The set of positive operators $\mathcal{B}(H)^+$ is strongly closed in $\mathcal{B}(H)$, and thus Borel.*

Proof. Part (a) is immediate from the weak continuity of the adjoint operation. Part (b) follows from the fact that the maps $T \mapsto \langle Tv, v \rangle$, for $v \in H$, are strongly continuous. \square

Proposition 4.5. *The set of projections $\mathcal{P}(H)$ is strongly closed in $\mathcal{B}(H)_{\leq 1}$, and thus Polish.*

Proof. Let $\mathcal{I} = \{T \in \mathcal{B}(H)_{\leq 1} : T^2 = T\}$. Since multiplication of operators $\mathcal{B}(H)_{\leq 1} \times \mathcal{B}(H)_{\leq 1} \rightarrow \mathcal{B}(H)$ is strongly continuous, it follows that \mathcal{I} is strongly closed in $\mathcal{B}(H)_{\leq 1}$. Let $\mathcal{S} = \{T \in \mathcal{B}(H)_{\leq 1} : T^* = T\}$, which is strongly closed in $\mathcal{B}(H)_{\leq 1}$. Clearly, $\mathcal{P}(H) = \mathcal{S} \cap \mathcal{I}$. \square

Most of the equivalence relations we study below arise from the ideals $\mathcal{B}_f(H)$, $\mathcal{K}(H)$ and $\mathcal{B}^p(H)$ for $1 \leq p < \infty$, and thus we will need to show that the corresponding ideal is Borel in the relevant topology.

Lemma 4.6. *In each of the norm, p -norm (for $1 \leq p < \infty$) and strong operator topologies, $\mathcal{B}_f(H)$ is separable.*

Proof. Approximate finite rank operators by matrices over $\mathbb{Q}(i)$. \square

Since $\mathcal{B}_f(H)$ is dense with respect to the relevant topologies, we have:

Proposition 4.7. (a) *$\mathcal{K}(H)$ is a separable Banach space with the operator norm. In particular, it is Polish.*

(b) *For $1 \leq p < \infty$, $\mathcal{B}^p(H)$ is a separable Banach space with the p -norm. In particular, it is Polish.*

Lemma 4.8. *For each $n \in \mathbb{N}$, the set $\mathcal{F}_{\leq n} = \{T \in \mathcal{B}(H) : \text{rank}(T) \leq n\}$ is strongly closed in $\mathcal{B}(H)$.*

Proof. ¹ Suppose that $T \in \mathcal{B}(H)$ is such that $\text{rank}(T) > n$. There are vectors $v_0, \dots, v_n \in H$ such that Tv_0, \dots, Tv_n are linearly independent, or equivalently, their Gram determinant $\det(\langle Tv_i, Tv_j \rangle_{i,j})$ is nonzero. Since the Gram determinant

¹We thank the anonymous referee for a much shortened proof of this fact.

is continuous, there is a strongly open neighborhood of T in $\mathcal{B}(H)$ such that for all S in that neighborhood, the Gram determinant $\det(\langle Sv_i, Sv_j \rangle_{i,j})$ is also nonzero, and so $\text{rank}(S) > n$. Thus, the complement of $\mathcal{F}_{\leq n}$ is strongly open. \square

The following proposition is an immediate consequence:

Proposition 4.9. $\mathcal{B}_f(H)$ is an F_σ subset of $\mathcal{B}(H)$ in the strong operator topology.

Proposition 4.10. $\mathcal{K}(H)$ is an $F_{\sigma\delta}$ subset of $\mathcal{B}(H)$ in the strong operator topology.

Proof. This is essentially the proof of the more general Theorem 3.1 in [4]. Let $\{T_k\}_{k=0}^\infty$ be a norm-dense sequence in $\mathcal{K}(\mathcal{H})$. Let $B = \mathcal{B}(\mathcal{H})_{\leq 1}$. Then,

$$\mathcal{K}(\mathcal{H}) = \bigcap_{n=0}^\infty \left(\mathcal{K}(\mathcal{H}) + \frac{1}{n+1} B \right) \supseteq \bigcap_{n=0}^\infty \bigcup_{k=0}^\infty \left(T_k + \frac{1}{n+1} B \right) \supseteq \mathcal{K}(\mathcal{H}).$$

B is strongly closed, and thus $\mathcal{K}(\mathcal{H})$ is $F_{\sigma\delta}$. \square

Proposition 4.11. For each $1 \leq p < \infty$, $\mathcal{B}^p(H)$ is a Polishable subspace of $\mathcal{K}(H)$ in the norm topology.

Proof. We have seen that $\mathcal{B}^p(H)$ is a separable Banach space under the p -norm. It remains to verify that the Borel structures in both topologies coincide. This follows easily from the fact that $\|T\| \leq \|T\|_p$ for $T \in \mathcal{B}^p(H)$, showing that the inclusion map $i : \mathcal{B}^p(H) \rightarrow \mathcal{K}(H)$ is a continuous injection from the p -norm to the norm topology. \square

Lemma 4.12 (cf. p. 48 of [5]). *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Borel function, then the map $\mathcal{B}(H)_{sa} \rightarrow \mathcal{B}(H)_{sa} : T \mapsto f(T)$ is Borel.*

Proof. Let $(p_n)_n$ be a sequence of real polynomials converging to f pointwise, which are uniformly bounded on compact sets. It follows by basic spectral theory that, for $T \in \mathcal{B}(H)_{sa}$, $p_n(T)$ converges to $f(T)$ weakly. Thus, f is a pointwise (weak) limit of Borel functions, and hence Borel. \square

Proposition 4.13. For each $1 \leq p < \infty$, $\mathcal{B}^p(H)$ is a Borel subset of $\mathcal{B}(H)$ in the strong operator topology.

Proof. Fix $1 \leq p < \infty$. Applying Lemma 4.12 to the map $T \mapsto |T|^p$, we see that this function is Borel, and observe that if $\{e_n : n \in \mathbb{N}\}$ is a fixed orthonormal basis for H , then

$$T \in \mathcal{B}^p(H) \quad \Leftrightarrow \quad \exists M \forall N \left(\sum_{n=0}^N \langle |T|^p e_n, e_n \rangle < M \right),$$

showing that $\mathcal{B}^p(H)$ is Borel. \square

5. EQUIVALENCE RELATIONS IN $\mathcal{B}(H)$

Throughout this section, $\mathcal{B}(H)$ will be considered as a standard Borel space with the Borel structure induced by the strong operator topology, $\mathcal{B}(H)_{\leq 1}$ a Polish space with the strong operator topology, and $\mathcal{K}(H)$ a Polish space with the norm topology. We will consider the equivalence relations on $\mathcal{B}(H)$, and their restrictions to $\mathcal{B}(H)_{\leq 1}$

and $\mathcal{K}(H)$, induced by the ideals $\mathcal{B}_f(H)$, $\mathcal{K}(H)$ and $\mathcal{B}^p(H)$ for $1 \leq p < \infty$, denoted (and named) as follows:

$$\begin{aligned} T \equiv_f S &\Leftrightarrow T - S \in \mathcal{B}_f(H) \quad (\text{modulo finite rank}) \\ T \equiv_{\text{ess}} S &\Leftrightarrow T - S \in \mathcal{K}(H) \quad (\text{modulo compact or essential equivalence}) \\ T \equiv_p S &\Leftrightarrow T - S \in \mathcal{B}^p(H) \quad (\text{modulo } p\text{-class}), \text{ for } 1 \leq p < \infty. \end{aligned}$$

Proposition 5.1. *The equivalence relations \equiv_f , \equiv_{ess} and \equiv_p for $1 \leq p < \infty$ are Borel in the standard Borel space $\mathcal{B}(H)$, and when restricted to the Polish spaces $\mathcal{B}(H)_{\leq 1}$ and $\mathcal{K}(H)$. In both of the latter, \equiv_f is F_σ , while \equiv_{ess} is $F_{\sigma\delta}$.*

Proof. This is immediate from Propositions 4.9, 4.10, 4.11 and 4.13. \square

Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for H for the remainder of this section. Consider the map $\ell^\infty \rightarrow \mathcal{B}(H) : \alpha \mapsto T_\alpha$ given by

$$T_\alpha v = \sum_{n=0}^{\infty} \alpha_n \langle v, e_n \rangle e_n,$$

for $\alpha = (\alpha_n)_n \in \ell^\infty$ and $v \in H$.

- Lemma 5.2.** (a) *The map $\ell^\infty \rightarrow \mathcal{B}(H) : \alpha \mapsto T_\alpha$ is an isometric embedding with respect to the usual norms on these spaces, and maps c_0 into $\mathcal{K}(H)$.*
(b) *The map $[0, 1]^\mathbb{N} \rightarrow \mathcal{B}(H) : \alpha \mapsto T_\alpha$ is continuous when $[0, 1]^\mathbb{N}$ is endowed with the product topology, and $\mathcal{B}(H)$ with the strong operator topology. Its range is contained within $\mathcal{B}(H)_{\leq 1}$.*

Proof. (a) This map is the well-known isometric embedding of ℓ^∞ as diagonal multiplication operators on H (cf. 4.7.6 in [17]). That it maps c_0 into $\mathcal{K}(H)$ is a restatement of Proposition 2.1.

(b) Fix $\alpha \in [0, 1]^\mathbb{N}$ and let $U = \{T \in \mathcal{B}(H) : \|(T - T_\alpha)v\| < \epsilon\}$, a subbasic open neighborhood of T_α in the strong operator topology, where $v = \sum_{n=0}^{\infty} a_n e_n$ and $\epsilon > 0$. Pick m such that $\sum_{n=m+1}^{\infty} |a_n|^2 < \epsilon^2/2$, and let

$$V = \left\{ \beta \in [0, 1]^\mathbb{N} : \sum_{n=0}^m |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2/2 \right\}.$$

V is an open neighborhood of α in $[0, 1]^\mathbb{N}$. If $\beta \in V$, then

$$\|(T_\beta - T_\alpha)v\|^2 = \sum_{n=0}^m |\beta_n - \alpha_n|^2 |a_n|^2 + \sum_{n=m+1}^{\infty} |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2,$$

showing that $T_\beta \in U$. It follows that the map is continuous. \square

Let $X = \prod_{n=0}^{\infty} [0, \frac{1}{n+1}]$, and consider the equivalence relation E :

$$\alpha E \beta \Leftrightarrow \exists m \forall n \geq m (\alpha_n = \beta_n)$$

for $\alpha, \beta \in X$. This can be identified (up to Borel bi-reducibility) with E_1 .

Proposition 5.3. *$E_1 \sqsubseteq_B \equiv_f$ restricted to $\mathcal{B}(H)_{\leq 1}$ and $\mathcal{K}(H)$. Thus, \equiv_f (in either space) is not Borel reducible to the orbit equivalence relation of any Polish group action.*

Proof. We use the restriction of the map $\alpha \mapsto T_\alpha$ to X . By Lemma 5.2, it is continuous in both relevant topologies. Moreover, $T_\alpha - T_\beta$ is of finite rank if and only if $\alpha E_1 \beta$. The remaining claim follows by Theorem 2.5. \square

Corollary 5.4. $\mathcal{B}_f(H)$ is not Polishable in either the norm or strong operator topologies.

Proof. If $\mathcal{B}_f(H)$ was Polishable in either topology, then its translation action on $\mathcal{K}(H)$ would be a Polish group action, contrary to the above. \square

Proposition 5.5. $[0, 1]^\mathbb{N}/c_0 \subseteq_B \equiv_{\text{ess}}$ restricted to $\mathcal{B}(H)_{\leq 1}$. Thus, \equiv_{ess} (on $\mathcal{B}(H)$ or $\mathcal{B}(H)_{\leq 1}$) is not classifiable by countable structures.

Proof. We use the map $[0, 1]^\mathbb{N} \rightarrow \mathcal{B}(H) : \alpha \mapsto T_\alpha$ defined above. Suppose that $\alpha, \beta \in [0, 1]^\mathbb{N}$, then

$$(T_\alpha - T_\beta)v = \sum_{n=0}^{\infty} (\alpha_n - \beta_n) \langle v, e_n \rangle e_n$$

for $v \in H$. By Theorem 2.1, $T_\alpha - T_\beta$ is compact if and only if $\alpha - \beta \in c_0$. The remaining claim follows by Theorem 2.6. \square

Proposition 5.6. For $1 \leq p < \infty$,

(a) $[0, 1]^\mathbb{N}/\ell^p \subseteq_B \equiv_p$ restricted to $\mathcal{B}(H)_{\leq 1}$.

(b) $c_0/\ell^p \subseteq_B \equiv_p$ restricted to $\mathcal{K}(H)$

Thus, \equiv_p (on $\mathcal{B}(H)$, restricted to $\mathcal{B}(H)_{\leq 1}$, or $\mathcal{K}(H)$) is not classifiable by countable structures.

Proof. We again use the map $[0, 1]^\mathbb{N} \rightarrow \mathcal{B}(H)_{\leq 1} : \alpha \mapsto T_\alpha$. Fix $1 \leq p < \infty$. Suppose that $\alpha, \beta \in [0, 1]^\mathbb{N}$. For $x \in H$, we have that

$$|T_\beta - T_\alpha|^p x = \sum_{n=0}^{\infty} |\beta_n - \alpha_n|^p \langle x, e_n \rangle e_n,$$

and so,

$$\text{tr}(|T_\beta - T_\alpha|^p) = \sum_{n=0}^{\infty} \langle |T_\alpha - T_\beta|^p e_n, e_n \rangle = \sum_{n=0}^{\infty} |\beta_n - \alpha_n|^p.$$

Thus, $\alpha - \beta \in \ell^p$ if and only if $T_\alpha - T_\beta \in \mathcal{B}^p(H)$, and the map is a reduction. The proof of (b) is identical except that it uses the restriction of the map $\alpha \mapsto T_\alpha$ to c_0 . The remaining claim follows again by Theorem 2.6. \square

6. EQUIVALENCE RELATIONS IN $\mathcal{P}(H)$

Recall that $\mathcal{P}(H)$ is the set of projections in $\mathcal{B}(H)$, a Polish space with the strong operator topology. Several equivalence relations arise naturally on $\mathcal{P}(H)$ from the correspondence between projections and closed subspaces.

Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for H throughout this section. For each $x \subseteq \mathbb{N}$, let P_x be the projection onto the subspace $\overline{\text{span}}\{e_n : n \in x\}$. For $v \in H$,

$$P_x v = \sum_{n \in x} \langle v, e_n \rangle e_n.$$

The map $x \mapsto P_x$ is called the *diagonal embedding* (with respect to this basis), and is the restriction to $2^\mathbb{N}$ of the map $\alpha \mapsto T_\alpha$ from §5. In particular, it is a continuous injection by Lemma 5.2.

6.1. Equivalence of rank. For projections P, Q in $\mathcal{P}(H)$, we write $P \cong Q$ if $\text{ran}(P)$ and $\text{ran}(Q)$ are isometrically isomorphic, and $P \approx Q$ if $P \cong Q$ and $P^\perp \cong Q^\perp$. It is easy to see that \cong coincides with *Murray-von Neumann equivalence* of projections, and \approx coincides with *unitary equivalence* (see Remark 4.1.5 and Exercise 4.7 in [15]).

The \cong -classes in $\mathcal{P}(H)$ are those given by rank, and so there are exactly \aleph_0 -many classes. Likewise, the \approx -classes are those given by rank and *co-rank* (the dimension of the kernel of the projection), of which there are also \aleph_0 -many classes. Denote these classes by

$$\mathcal{P}_n(H) = \{P \in \mathcal{P}(H) : \text{rank}(P) = n\},$$

and

$$\mathcal{P}_{n,k}(H) = \{P \in \mathcal{P}(H) : \text{rank}(P) = n \text{ and } \text{corank}(P) = k\},$$

respectively, for $n, k \in \mathbb{N} \cup \{\infty\}$.

Lemma 6.1. (a) For each $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{P}_n(H)$ is G_δ .

(b) For each $n, k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{P}_{n,k}(H)$ is G_δ .

Proof. (a) For $n \in \mathbb{N}$, with $n \geq 1$, $\mathcal{P}_n(H) = \mathcal{F}_{\leq n} \cap (\mathcal{P}(H) \setminus \mathcal{F}_{\leq n-1})$, the intersection of a closed set (by Lemma 4.8) and an open set, and thus is G_δ . $\mathcal{P}_\infty(H)$ is the complement of the F_σ set $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{\leq n} \cap \mathcal{P}(H)$.

(b) Follows from (a), noting that $\mathcal{P}_{n,k}(H) = \mathcal{P}_n(H) \cap \mathcal{P}_k(H)^\perp$ and that the map $P \mapsto P^\perp = I - P$ is a self-homeomorphism of $\mathcal{P}(H)$. \square

Proposition 6.2. The relations \cong and \approx are Borel equivalence relations, and Borel bireducible with $\Delta(\mathbb{N})$.

Proof. The relations are Borel, being unions of countably many classes, each Borel by the previous lemma. Likewise, the functions $\mathcal{P}(H) \rightarrow \mathbb{N} \cup \{\infty\} : P \mapsto \text{rank}(P)$ and $\mathcal{P}(H) \rightarrow (\mathbb{N} \cup \{\infty\})^2 : P \mapsto (\text{rank}(P), \text{corank}(P))$ are Borel reductions to $\Delta(\mathbb{N} \cup \{\infty\})$, which is clearly bireducible with $\Delta(\mathbb{N})$. \square

“Most” projections are of infinite rank and co-rank:

Proposition 6.3. $\mathcal{P}_{\infty, \infty}(H)$ is a dense G_δ subset of $\mathcal{P}(H)$.

Proof. It suffices to show that $\mathcal{P}_{\infty, \infty}(H)$ is dense. Note that every non-empty open neighborhood in $2^\mathbb{N}$ contains a $y \subseteq \mathbb{N}$ which is infinite and co-infinite. Let $P \in \mathcal{P}(H)$, $U \subseteq \mathcal{P}(H)$ be an open neighborhood of P , and $\{f_k : k \in \mathbb{N}\}$ be an orthonormal basis for which P is diagonal. Since the diagonal embedding $2^\mathbb{N} \rightarrow \mathcal{P}(H) : x \mapsto P_x$ relative to $\{f_k : k \in \mathbb{N}\}$ is continuous, we can pick a $y \in 2^\mathbb{N}$ infinite and co-infinite in \mathbb{N} , and contained in the inverse image of U under the diagonal map. Then, $P_y \in U$ is a projection of infinite rank and infinite co-rank. \square

6.2. Equivalence modulo finite rank. There are two natural ways to define equivalence modulo *finite rank* or *finite dimension* on $\mathcal{P}(H)$. One could simply restrict \equiv_f to $\mathcal{P}(H)$, or one could say that $P \equiv_{fd} Q$ if there exist finite dimensional subspaces U and V of H such that $\text{ran}(P) \subseteq \text{ran}(Q) + U$ and $\text{ran}(Q) \subseteq \text{ran}(P) + V$. In fact, these notions coincide. We will use the fact that if V is a closed subspace of H and F a finite dimensional subspace of H , then $V + F$ is closed (E 2.1.4 in [17]).

Proposition 6.4. Let $P, Q \in \mathcal{P}(H)$. The following are equivalent:

- (i) $P \equiv_{fd} Q$.
- (ii) *There exist finite dimensional subspaces $W \subseteq \text{ran}(P)^\perp$ and $Y \subseteq \text{ran}(Q)^\perp$ such that $\text{ran}(P) + W = \text{ran}(Q) + Y$.*
- (iii) $P \equiv_f Q$.

Proof. (i) \Rightarrow (ii): Let U and V witness $P \equiv_{fd} Q$ as in the definition, say with $\{u_0, \dots, u_m\}$ a basis for U , and $\{v_0, \dots, v_n\}$ a basis for V . Note that $\text{ran}(Q) + U + V = \text{ran}(P) + U + V$. If $w \in \{u_0, \dots, u_m, v_0, \dots, v_n\}$, we can write $w = p + p^\perp$ where $p \in \text{ran}(P)$ and $p^\perp \in \text{ran}(P)^\perp$. Collect the p^\perp for each such w as $\{w_0, \dots, w_k\}$. Then, for $W = \text{span}\{w_0, \dots, w_k\}$, we have

$$\text{ran}(P) + U + V = \text{ran}(P) + W.$$

Likewise, we can find a finite dimensional subspace $Y \subseteq \text{ran}(Q)^\perp$ with

$$\text{ran}(Q) + U + V = \text{ran}(Q) + Y.$$

(ii) \Rightarrow (iii): For W and Y as in (ii), let R be the projection onto W and R' the projection onto Y . Since W is orthogonal to $\text{ran}(P)$, $P + R$ is the projection onto $\text{ran}(P) + W$. Likewise $Q + R'$ is the projection onto $\text{ran}(Q) + Y$. Thus, $P + R = Q + R'$, and so $P - Q = R' - R$, a finite rank operator.

(iii) \Rightarrow (i): Suppose that $P - Q = A$ where $A \in \mathcal{B}_f(A)$. Then,

$$\text{ran}(P) = \overline{\text{ran}(Q + A)} \subseteq \overline{\text{ran}(Q) + \text{ran}(A)} = \text{ran}(Q) + \text{ran}(A),$$

and likewise,

$$\text{ran}(Q) = \overline{\text{ran}(P - A)} \subseteq \overline{\text{ran}(P) + \text{ran}(A)} = \text{ran}(P) + \text{ran}(A).$$

Since $\text{ran}(A)$ is finite-dimensional, it follows that $P \equiv_{fd} Q$. \square

Consequently, we will use \equiv_f for this (Borel) equivalence relation on $\mathcal{P}(H)$. It is easy to see that the diagonal embedding witnesses the non-smoothness of \equiv_f on $\mathcal{P}(H)$: Given $x, y \in 2^\mathbb{N}$,

$$(P_x - P_y)v = \sum_{n=0}^{\infty} (x_n - y_n) \langle v, e_n \rangle e_n$$

for all $v \in H$, and this operator is finite rank if and only if all but finitely many of the terms $x_n - y_n$ are 0.

To show that \equiv_f restricted to $\mathcal{P}(H)$ is of higher complexity, we define a new map $[0, 1]^\mathbb{N} \rightarrow \mathcal{P}(H) : \alpha \mapsto P_\alpha$ as follows: For each $\alpha = (\alpha_n)_n \in [0, 1]^\mathbb{N}$, let P_α be the projection onto $\overline{\text{span}}\{e_{2n} + \alpha_n e_{2n+1} : n \in \mathbb{N}\}$. This is the first map into the space of operators that we have considered whose range is not simultaneously diagonalizable by some basis, nor even commutative.

Lemma 6.5. *The map $[0, 1]^\mathbb{N} \rightarrow \mathcal{P}(H) : \alpha \mapsto P_\alpha$ is a continuous injection.*

Proof. First we show that $\alpha \mapsto P_\alpha$ is injective. Let $\alpha, \beta \in [0, 1]^\mathbb{N}$ with $\alpha \neq \beta$, so $\alpha_k \neq \beta_k$ for some k . In order to show that $P_\alpha \neq P_\beta$, it suffices to show that $P_\alpha(e_{2k} + \beta_k e_{2k+1}) \neq e_{2k} + \beta_k e_{2k+1} = P_\beta(e_{2k} + \beta_k e_{2k+1})$. Note that

$$P_\alpha(e_{2k} + \beta_k e_{2k+1}) = \frac{1 + \alpha_k \beta_k}{1 + \alpha_k^2} (e_{2k} + \alpha_k e_{2k+1}).$$

But, by linear independence of e_{2k} and e_{2k+1} ,

$$\frac{1 + \alpha_k \beta_k}{1 + \alpha_k^2} (e_{2k} + \alpha_k e_{2k+1}) = e_{2k} + \beta_k e_{2k+1} \quad \Leftrightarrow \quad \alpha_k = \beta_k,$$

contrary to our assumption. Thus, $P_\alpha \neq P_\beta$.

To see that the map is continuous,² for each $n \in \mathbb{N}$ and $\alpha \in [0, 1]^\mathbb{N}$, let $P_{n,\alpha}$ be the projection of H onto $\text{span}\{e_{2n} + \alpha_n e_{2n+1}\}$. It is clear that for each n , the map $[0, 1]^\mathbb{N} \rightarrow \mathcal{P}(H) : \alpha \mapsto P_{n,\alpha}$ is strongly continuous. Moreover, $P_\alpha = \bigoplus_{n \in \mathbb{N}} P_{n,\alpha}$. Strong continuity of the map $\alpha \mapsto P_\alpha$ follows easily from these facts, and that $\|P_{n,\alpha}\| \leq 1$ for all n and α . \square

For $\alpha \in [0, 1]^\mathbb{N}$, the vectors $\frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1})$, $n \in \mathbb{N}$, form an orthonormal basis for $\text{ran}(P_\alpha)$. Thus we can write, for $v = \sum_{n=0}^\infty a_n e_n \in H$,

$$\begin{aligned} P_\alpha v &= \sum_{n=0}^\infty \left\langle \sum_{k=0}^\infty a_k e_k, \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \right\rangle \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \\ &= \sum_{n=0}^\infty \frac{a_{2n} + a_{2n+1}\alpha_n}{1 + \alpha_n^2} (e_{2n} + \alpha_n e_{2n+1}). \end{aligned}$$

Since we must consider the operator $P_\alpha - P_\beta$ several times in what follows, it will be useful to have it in a canonical form. For $v = \sum_{n=0}^\infty a_n e_n$,

$$\begin{aligned} (P_\alpha - P_\beta)v &= \sum_{n=0}^\infty \left[\frac{a_{2n} + a_{2n+1}\alpha_n}{1 + \alpha_n^2} - \frac{a_{2n} + a_{2n+1}\beta_n}{1 + \beta_n^2} \right] e_{2n} \\ &\quad + \sum_{n=0}^\infty \left[\frac{a_{2n}\alpha_n + a_{2n+1}\alpha_n^2}{1 + \alpha_n^2} - \frac{a_{2n}\beta_n + a_{2n+1}\beta_n^2}{1 + \beta_n^2} \right] e_{2n+1}. \end{aligned}$$

Denote by T_0, T_1, T_2 and T_3 the operators

$$\begin{aligned} T_0 v &= \sum_{n=0}^\infty \left[\frac{1}{1 + \alpha_n^2} - \frac{1}{1 + \beta_n^2} \right] a_{2n} e_{2n}, \\ T_1 v &= \sum_{n=0}^\infty \left[\frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right] a_{2n+1} e_{2n+1}, \\ T_2 v &= \sum_{n=0}^\infty \left[\frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right] a_{2n} e_{2n}, \\ T_3 v &= \sum_{n=0}^\infty \left[\frac{\alpha_n^2}{1 + \alpha_n^2} - \frac{\beta_n^2}{1 + \beta_n^2} \right] a_{2n+1} e_{2n+1}, \end{aligned}$$

and by S_0 and S_1 the operators

$$S_0 v = \sum_{n=0}^\infty a_{2n+1} e_{2n} \quad \text{and} \quad S_1 v = \sum_{n=0}^\infty a_{2n} e_{2n+1}.$$

Each of the aforementioned operators is bounded, and by collecting terms, one can show that

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3.$$

²We thank the anonymous referee for a much shortened proof of this fact.

Theorem 6.6. $E_1 \sqsubseteq_B \equiv_f$ restricted to $\mathcal{P}(H)$. Thus, \equiv_f restricted to $\mathcal{P}(H)$ is not Borel reducible to the orbit equivalence relation of any Polish group action.³

Proof. Represent E_1 on $[0, 1]^\mathbb{N}$ by $\alpha E_1 \beta \Leftrightarrow \exists m \forall n \geq m (\alpha_n = \beta_n)$. As above, for $\alpha, \beta \in [0, 1]^\mathbb{N}$, we have the representation $P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3$. Clearly, if $\alpha E_1 \beta$, then all but finitely many of the coefficients (which are independent of v) $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right]$, $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right]$ and $\left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right]$ will be 0, showing that $P_\alpha - P_\beta$ has finite rank.

Conversely, suppose that $P_\alpha - P_\beta$ has finite rank. It follows that the operator $T = T_0 + S_0 T_1$, given by

$$Tv = \sum_{n=0}^{\infty} \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right] a_{2n} e_{2n} + \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] a_{2n+1} e_{2n}$$

for $v = \sum_{n=0}^{\infty} a_n e_n$, is of finite rank. Using vectors of the form $\sum_{n=0}^{\infty} a_{2n} e_{2n}$ and $\sum_{n=0}^{\infty} a_{2n+1} e_{2n+1}$ it is easy to see that in order for T to be finite rank, all but finitely many of the terms $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right]$, and $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right]$ are 0. Since $\alpha_n \geq 0$ and $\beta_n \geq 0$, $\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} = 0$ if and only if $\alpha_n = \beta_n$. Thus, $\alpha E_1 \beta$, and the map is a reduction. The rest follows by Theorem 2.5. \square

6.3. Essential equivalence. The last equivalence relation we wish to study is the restriction of \equiv_{ess} to $\mathcal{P}(H)$. The quotient of $\mathcal{P}(H)$ by this relation can be identified with the set of projections in Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$, by Proposition 3.1 in [19].

We note that, although a projection is compact if and only if it is of finite rank, this is not true of the difference of two projections. In particular, \equiv_{ess} does not coincide with \equiv_f on $\mathcal{P}(H)$. However, the same argument used above to show that $E_0 \sqsubseteq_B \equiv_f$, also shows that $E_0 \sqsubseteq_B \equiv_{\text{ess}}$.

Our remaining goal is to show that $[0, 1]^\mathbb{N}/c_0$ is Borel reducible to \equiv_{ess} on $\mathcal{P}(H)$, and thus, the latter is not classifiable by countable structures. We will again use the map $[0, 1]^\mathbb{N} \rightarrow \mathcal{P}(H) : \alpha \mapsto P_\alpha$ used to prove Theorem 6.6. As in the previous section, we have the representation

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3,$$

when $\alpha, \beta \in [0, 1]^\mathbb{N}$.

Theorem 6.7. $[0, 1]^\mathbb{N}/c_0 \sqsubseteq_B \equiv_{\text{ess}}$ restricted to $\mathcal{P}(H)$. Thus, \equiv_{ess} restricted to $\mathcal{P}(H)$ is not classifiable by countable structures.

Proof. We have seen in Lemma 6.5 that the map $\alpha \mapsto P_\alpha$ is a continuous injection. It remains to show that this map is a reduction of $[0, 1]^\mathbb{N}/c_0$ to \equiv_{ess} . Let $\alpha, \beta \in [0, 1]^\mathbb{N}$, and suppose that $\alpha - \beta \in c_0$. Using Proposition 2.1, and the inequalities

$$\left| \frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right| = \left| \frac{\beta_n^2 - \alpha_n^2}{(1+\alpha_n^2)(1+\beta_n^2)} \right| \leq |\beta_n - \alpha_n| |\beta_n + \alpha_n|,$$

³The author is indebted to Ilijas Farah for suggesting this result and ideas of its proof.

$$\begin{aligned}
\left| \frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right| &= \left| \frac{\alpha_n + \alpha_n \beta_n^2 - \beta_n - \alpha_n^2 \beta_n}{(1 + \alpha_n^2)(1 + \beta_n^2)} \right| \\
&\leq |\alpha_n - \beta_n| + |\alpha_n| |\beta_n - \alpha_n| |\beta_n|, \\
\left| \frac{\alpha_n^2}{1 + \alpha_n^2} - \frac{\beta_n^2}{1 + \beta_n^2} \right| &= \left| \frac{\alpha_n^2 - \beta_n^2}{(1 + \alpha_n^2)(1 + \beta_n^2)} \right| \leq |\beta_n - \alpha_n| |\beta_n + \alpha_n|,
\end{aligned}$$

we have that T_0, T_1, T_2 and T_3 are compact. Since the compact operators form an ideal, $S_0 T_1$ and $S_1 T_2$ are also compact, and thus so is $P_\alpha - P_\beta$.

Conversely, suppose that $P_\alpha - P_\beta$ is compact. We will use that if an operator is compact, then it is weak-norm continuous on the closed unit ball of H (3.3.3 in [17]). Since the sequence e_m converges *weakly* to 0 as $m \rightarrow \infty$, it follows that $(P_\alpha - P_\beta)e_{2m}$ and $(P_\alpha - P_\beta)e_{2m+1}$ converge *in norm* to 0. Observe that

$$\begin{aligned}
(P_\alpha - P_\beta)e_{2m} &= \left[\frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right] e_{2m} + \left[\frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right] e_{2m+1}, \\
(P_\alpha - P_\beta)e_{2m+1} &= \left[\frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right] e_{2m} + \left[\frac{\alpha_m^2}{1 + \alpha_m^2} - \frac{\beta_m^2}{1 + \beta_m^2} \right] e_{2m+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(P_\alpha - P_\beta)e_{2m}\|^2 &= \left| \frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right|^2 + \left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right|^2, \\
\|(P_\alpha - P_\beta)e_{2m+1}\|^2 &= \left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right|^2 + \left| \frac{\alpha_m^2}{1 + \alpha_m^2} - \frac{\beta_m^2}{1 + \beta_m^2} \right|^2
\end{aligned}$$

and both converge to 0 as $m \rightarrow \infty$. Using the inequalities

$$\begin{aligned}
\left| \frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right| &= \left| \frac{\beta_m^2 - \alpha_m^2}{(1 + \alpha_m^2)(1 + \beta_m^2)} \right| \geq \frac{1}{4} |\alpha_m - \beta_m| |\alpha_m + \beta_m|, \\
\left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right| &= \left| \frac{\alpha_m + \alpha_m \beta_m^2 - \beta_m - \alpha_m^2 \beta_m}{(1 + \alpha_m^2)(1 + \beta_m^2)} \right| \\
&\geq \frac{1}{4} |\alpha_m - \beta_m| |1 - \alpha_m \beta_m|,
\end{aligned}$$

we have that the quantities on the right must also converge to 0. For any m , since $\alpha_m, \beta_m \in [0, 1]$, we have $\alpha_m + \beta_m \geq \sqrt{2\alpha_m \beta_m} \geq \alpha_m \beta_m$ and so

$$|\alpha_m + \beta_m| + |1 - \alpha_m \beta_m| = \alpha_m + \beta_m + 1 - \alpha_m \beta_m \geq 1.$$

Thus,

$$|\alpha_m - \beta_m| |\alpha_m + \beta_m| + |\alpha_m - \beta_m| |1 - \alpha_m \beta_m| \geq |\alpha_m - \beta_m|,$$

showing that $\alpha_m - \beta_m$ converges to 0, i.e., $\alpha - \beta \in c_0$, as claimed. Non-classifiability follows by Theorem 2.6. \square

7. FURTHER QUESTIONS

We have seen that the equivalence relations $[0, 1]^{\mathbb{N}}/c_0$ and $[0, 1]^{\mathbb{N}}/\ell^p$ (or $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell^p$) are Borel reducible to \equiv_{ess} and \equiv_p for $1 \leq p < \infty$, respectively. We may think of \equiv_{ess} and \equiv_p as non-commutative analogues of $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell^p$, and ask whether they are of the same complexity:

Question. Are the equivalence relations \equiv_{ess} and \equiv_p on $\mathcal{B}(H)$ Borel reducible to $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell^p$ for $1 \leq p < \infty$, respectively? Likewise for the restriction of \equiv_{ess} to $\mathcal{P}(H)$.

The Weyl–von Neumann theorem and the work of Ando–Matsuzawa [1] show that unitary equivalence modulo compact on bounded self-adjoint operators is smooth. The refinement of this given by unitary equivalence modulo Schatten p -class has also been studied; see the classification theorem of Carey and Pincus in [2]. In light of this work, we ask:

Question. What is the Borel complexity of unitary equivalence modulo Schatten p -class? Is it smooth? Is it classifiable by countable structures?

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